数理解析研究所講究録
第 1708 巻 2010 年 46-49

On weak notion of $\mathfrak{p}$−dividing

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Abstract

I considered the restricted notions of weak dividing. In this note, I try to define a weak notion of $\mathfrak{p}$−dividing (thorn−dividing).

1. Preliminaries

We recall some definitions.

Definition 1 Let $\varphi(x_0, x_1, \cdots\cdots, x_{n-1})$ be a formula and $p(x)$ be a type. We denote the type $\{\varphi(x_0, x_1, \cdots\cdots, x_{n-1})\} \cup p(x_0) \cup p(x_1) \cup \cdots\cdots \cup p(x_{n-1})$ by $[p]^{\varphi}$.

Let $A \subset B$ and $p(x) \in S(B)$.

$p(x)$ *divides overA* if there is a formula $\varphi(x, b) \in p(x)$ and an infinite sequence $\{b_i : i < \omega\}$ with $b \equiv b_i(A)$ such that $\{\varphi(x, b_i) : i < \omega\}$ is k−inconsistent for some $k < \omega$.

$p(x)$ *weakly divides over* A if there is a formula $\varphi(\bar{x}) \in L_n(A)$ such that $[p\lceil A]^{\varphi}$ is consistent, while $[p]^{\varphi}$ is inconsistent.

We can define weak dividing for formulas.

Let $b \notin A$.

$\psi(x, b)$ *weakly divides overA* if there is a formula $\varphi(\bar{x}) \in L_n(A)$ and a realization a of $\psi(x, b)$ such that $[\mathrm{tp}(a/A)]^{\varphi}$ is consistent, while $[\psi(x, b)]^{\varphi}$ is inconsistent.

And we can consider weak forking.

$p(x)$ *weakly forks over* A if there is a $q(x, y) \in S(A)$ such that $p(x) \cup q(x, y)$ is consistent, and any completion $r(x, y) \in S(B)$ of $p(x) \cup q(x, y)$ weakly divides over A.

If we exchange the role between variables and parametes in the definition of weak dividing, we could define weak forking naturally.

In this note, we call such formula "$\varphi(\bar{x})$" in the definition above the *witness formula* of weak dividing for the sake of convenience.

I introduce an example from [3].

Example 2 Let T be the theory of an equivalence relation with two infinite classes of the language $L = \{\text{a binary relation } E(x, y)\}$. And let $\models \neg E(a, b)$. Then the type $\mathrm{tp}(a/b)$ does not divide over $\emptyset$, while $\mathrm{tp}(a/b)$ weakly divides over $\emptyset$ by the formula $\neg E(x, y)$.

I tried to divide witness formulas into some classes according to their properties ago. And I told about the next characterization at the RIMS meeting last year.

Definition 3 Let $A \subset B$ and $p(x) \in S(B)$.
$p(x)$ *$\mathcal{M}-$weakly divides over A* if there is a formula $\varphi(\bar{x}) \in L_n(A)$ and a Morley sequence $I = \{a_i : i < n+1\}$ of $p\lceil A$ such that $\models \varphi(a_0, a_1, \cdots\cdots, a_{n-1})$, while the type $[p]^{\varphi}$ is inconsistent.

Theorem 4 *Let T be simple.*
Then T is stable if and only if $\mathcal{M}-$weak dividing over models is symmetric.

2. Weak notion of $\mathfrak{p}-$dividing

In recent years another variant of dividing, "*thorn*"$-$dividing has been characterized in rosy theory (see e.g. [4]). I tried to define weak notion of $\mathfrak{p}-$dividing (thorn$-$dividing). We recall some definitions first.

Definition 5 Let $A \subset B$ and $p(x) \in S(B)$.
$p(x)$ *strongly divides over A* if there is a formula $\varphi(x, b) \in p(x)$ such that $b \notin acl(A)$ and $\{\varphi(x, b_i) : b_i \models \mathrm{tp}(b/A)\}$ is $k-$inconsistent for some $k < \omega$.
$p(x)$ $\mathfrak{p}$*$-$divides over A* if $p(x)$ strongly divides over Ac for some parameter c.
$p(x)$ $\mathfrak{p}$*$-$forks over A* if there is a formula $\varphi(x, b) \in p(x)$ such that $\varphi(x, b)$ implies a finite disjunction of formulas which $\mathfrak{p}$-divides over A.
Given a formula φ, a set Δ of formulas in variables x, y, a set of formulas Π in variables y, z, and a number k, we define $\mathfrak{p}(\varphi, \Delta, \Pi, k)$(thorn$-$rank) inductively as follows :
(1) $\mathfrak{p}(\varphi, \Delta, \Pi, k) \geq 0,\ \infty,\ \lambda$ for limit ordinal λ is defined as usual.
(2) $\mathfrak{p}(\varphi, \Delta, \Pi, k) \geq \alpha + 1$ if and only if there is a $\delta \in \Delta$, some $\pi(y, z) \in \Pi$ and parameters c such that
(a) $\mathfrak{p}(\varphi \wedge \delta(x, a), \Delta, \Pi, k) \geq \alpha$ for infinitely many $a \models \pi(y, c)$
(b) $\{\delta(x, a)\}_{a \models \pi(y,c)}$ is $k-$inconsistent.
For a type p, we define $\mathfrak{p}(p, \Delta, \Pi, k) = \min\{\mathfrak{p}(\varphi, \Delta, \Pi, k) \mid \varphi \in p\}$.
A theory T is *rosy* if for any type $p(x)$, any finite sets of formulas Δ and Π, and any finite k, $\mathfrak{p}(\varphi, \Delta, \Pi, k)$ is finite.

Remark 6 (1) In rosy theories, $\mathfrak{p}-$forking satisfies the independence axioms.

(2) If $a \models \varphi(x,b)$ and $\varphi(x,b)$ $\mathfrak{p}$−divides over C by the set $\{b_i \models \theta(y,d)\}$, then $b \in \mathrm{acl}(Cda) - \mathrm{acl}(Cd)$.

Weak notions of $\mathfrak{p}$−dividing could be defined in many ways. By the definition, $\mathfrak{p}$−dividing implies dividing. So we expect that weak $\mathfrak{p}$−dividing implies weak dividing.

Definition 7 Let $b \notin A$.
$\psi(x,b)$ *weakly* $\mathfrak{p}$−*divides over* A if there is a formula $\varphi(\bar{x}) = \exists y \bigwedge_{i<n} \theta(x_i, y)$ $\in L_n(A)$ and a realization a of $\psi(x,b)$ such that $[\mathrm{tp}(a/A)]^{\varphi}$ is consistent, while $[\psi(x,b)]^{\varphi}$ is inconsistent.

We define weak $\mathfrak{p}$−dividing($\mathfrak{p}$−forking) for types just like weak dividing(forking).

We can check the next fact easily.

Fact 8 *Let T be rosy. Then* $\mathfrak{p}$−*forking implies weak* $\mathfrak{p}$−*forking.*

3. Weak $\mathfrak{p}$−dividing and NIP theories

Definition 9 A formula $\varphi(x,y)$ has the *independence property* if for every $n < \omega$, there are sequences a_l $(l < n)$ such that for every $w \subset n$, $\models (\exists x)\left[\bigwedge_{l<n} \varphi(x,a_l)^{if\,(l \in w)}\right]$.

A theory T is *NIP* if no formula $\varphi(x,y)$ has the independence property.

Weak $\mathfrak{p}$−dividing is a kind of algebraic extension.

Lemma 10 (*T is any theory.*) $A \subset B$.
Then $\mathrm{tp}(a/B)$ *does not weakly* $\mathfrak{p}$−*divide over* A
if and only if
for any $n < \omega$, any C and any extension $q(x,C,A)$ of $\mathrm{tp}(a/A)$ *over AC, if $\bigcup_{i<n} q(x_i,C,A)$ is consistent, then $\bigcup_{i<n} q(x_i,Z,A) \cup \bigcup_{i<n} r(x_i,Y,A)$ is consistent where* $\mathrm{tp}(a/B) := r(x,B,A)$.

By the lemma above, we can prove the next fact.

Proposition 11 *Let T be NIP and unstable.*
Then weak $\mathfrak{p}$−*dividing is not symmetric.*

References

[1] S.Shelah, Simple unstable theories, Annals of Pure and Applied Logic 19 (1980) 177-203
[2] A.Dolich, Weak dividing, chain conditions, and simplicity, Archive for Mathematical Logic 43 (2004) 265-283
[3] B.Kim and N.Shi, A note on weak dividing, preprint
[4] A.Onshuus, Properties and consequences of Thorn−independence, Journal of Symbolic Logic 71 (2006) 1-21
[5] E.Hrushovski and A.Pillay, On NIP and invariant measures, preprint
[6] F.O.Wagner, Simple theories, Kluwer Academic Publishers (2000)